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Stability of a Fourth-Order Family of Iterative Methods for Solving Nonlinear Problems

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Abstract

In this paper a class of fourth-order methods for solving nonlinear equations and systems is presented. Some dynamical aspects of the family will be analyzed to determine the most stable members of the class. Moreover, a numerical estimation of the solution of Burgers' equation by using different elements of the family, is proposed.

Keywords: nonlinear equation, iterative methods, basins of attraction, Burgers' equation.

1 Introduction

Problems in science and engineering usually involve nonlinear equations or systems. The analytical solution of these kind of problems is difficult and, sometimes, we must use iterative methods in order to estimate the solutions. In fact, we will use the designed schemes to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. Moreover, the procedures will be extended for solving systems of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$.

In the scalar case, the efficiency of an iterative scheme for solving nonlinear equations is usually measured by means of the Efficiency Index, defined by Ostrowski in [8] as $I = p^{\frac{1}{d}}$, where p is the order of convergence of the method and d is the number of functional evaluations per step. In order to get optimal schemes, in the sense of Kung-Traub's conjecture [7], we must draw on multipoint iterative schemes. Many of them are very useful for solving nonlinear equations but they are not applicable to nonlinear systems.

In this work, we present a family of uniparametric two-point iterative procedure for solving the nonlinear equation $f(x)=0$. Its iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \left(1 - \frac{3}{4}\frac{u_k(1+\beta u_k)}{1+u_k(\beta+\frac{3}{2})}\right)\frac{f(x_k)}{f'(x_k)}, \end{aligned} \tag{1}$$

where $u_k = \frac{f'(y_k)-f'(x_k)}{f'(x_k)}$ and β is an arbitrary parameter. Let us note that this family contains known methods as Jarratt's scheme [6] (for $\beta = 0$). We prove that the local order of convergence of the elements of the family is four and so, all of them are optimal methods. We can also extend this family for solving nonlinear systems $F(x) = 0$, holding the order of convergence.

We will compare the proposed schemes with the well-known Newton's method, whose iterative expression is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots.$$

It is known that this scheme converges quadratically in some neighborhood of α, under standard conditions.

From the stability point of view, the behavior of the members of the family is analyzed on some test functions in the numerical section. The aim will be to choose the best elements of the class, in terms of stability and reliability. Finally, an applied multivariate problem will be solved, that is, the approximate solution of Burgers' equation, by using those more stable methods found by using dynamical tools.

2 Convergence analysis

In the following result the local order of convergence of the proposed class of methods (1) is analyzed, showing that, under the standard conditions, fourth-order of convergence is reached for any real value of the parameter β.

Theorem 2.1 *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function f:$D \in \mathbb{R} \to \mathbb{R}$ in a convex set D. For any real value of β, the scheme defined in* (1) *reaches fourth order of convergence, being its error equation*

$$e_{k+1} = \left(\left(1 - \frac{8}{3}\beta\right)c_2^3 - c_3c_2 + \frac{1}{9}c_4\right)e_k^4 + O(e_k^5), \tag{2}$$

where $c_k = (1/k!)\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 2, 3, 4, \ldots$, and $e_k = x_k - \alpha$.

Proof: By using Taylor expansion of $f(x_k)$ and $f'(x_k)$ around α, we obtain

$$f(x_k) = f'(\alpha)\left[e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + c_5e_k^5\right] + O(e_k^6) \tag{3}$$

and

$$f'(x_k) = f'(\alpha)\left[1 + 2c_2e_k + 3c_3{e_k}^2 + 4c_4{e_k}^3\right] + O(e_k^4). \tag{4}$$

From (3) and (4), we calculate the Taylor expansion of the quotient in the first step

$$\begin{aligned}\frac{f(x_k)}{f'(x_k)} &= e_k - c_2{e_k}^2 + 2(c_2^2 - c_3){e_k}^3 \\ &\quad +(-4c_2^3 + 4c_2c_3 + 3c_3c_2 - 3c_4){e_k}^4 + O({e_k}^5).\end{aligned}$$

Therefore, the error in the first step of the method is

$$\begin{aligned}y_k - \alpha &= \frac{1}{3}e_k + \frac{2}{3}c_2{e_k}^2 - \frac{4}{3}(c_2^2 - c_3){e_k}^3 \\ &\quad +\frac{2}{3}(4c_2^3 - 7c_2c_3 + 3c_4){e_k}^4 + O({e_k}^5).\end{aligned}$$

Furthermore,

$$\begin{aligned}f'(y_k) &= f'(\alpha)\left[1 + 2c_2(y_k-\alpha) + 3c_3(y_k-\alpha)^2 + 4c_4(y_k-\alpha)^3\right] + O((y_k-\alpha)^4) \\ &= f'(\alpha)\left[1 + \frac{2}{3}c_2e_k + \left(\frac{4}{3}c_2^2 + \frac{1}{3}c_3\right){e_k}^2 + \left(-\frac{8}{3}c_2^3 + 4c_2c_3 + \frac{4}{27}c_4\right){e_k}^3\right] + O({e_k}^4).\end{aligned} \tag{5}$$

Now, we get the Taylor expansion of u_k by using (4) and (5):

$$\begin{aligned}u_k &= \frac{f'(y_k) - f'(x_k)}{f'(x_k)} \\ &= -\frac{4}{3}c_2e_k + \left(4c_2^2 - \frac{8}{3}c_3\right){e_k}^2 + \left(-\frac{32}{3}c_2^3 + \frac{40}{3}c_2c_3 - \frac{104}{27}c_4\right){e_k}^3 + O({e_k}^4).\end{aligned}$$

So, we obtain

$$\frac{3}{4}\frac{u_k(1+\beta u_k)}{1+u_k(\beta+\frac{3}{2})} = -c_2e_k + (c_2^2 - 2c_3){e_k}^2 + (-\frac{8}{3}\beta c_2^3 + 2c_2c_3 - \frac{26}{9}c_4){e_k}^3 + O({e_k}^4)$$

and hence,

$$\left(1 - \frac{3}{4}\frac{u_k(1+\beta u_k)}{1+u_k(\beta+\frac{3}{2})}\right)\frac{f(x_k)}{f'(x_k)} = e_k + \left(\left(-1 + \frac{8}{3}\beta\right)c_2^3 + c_3c_2 - \frac{1}{9}c_4\right)e_k^4 + O({e_k}^5).$$

Finally, we have the final error equation of the method

$$e_{k+1} = \left(\left(1 - \frac{8}{3}\beta\right)c_2^3 - c_3c_2 + \frac{1}{9}c_4\right)e_k^4 + O(e_k^5)$$

$\square$

Let us remark that different methods can be obtained from (1) by using different values of β, some of them known ones. For example,

- if we use $\beta = 0$, classical Jarratt's scheme appears (see [6]), which second step in the iterative expression is

$$x_{k+1} = y_k - \frac{1}{2}\frac{f'(x_k)+3f'(y_k)}{3f'(y_k)-f'(x_k)}\frac{f(x_k)}{f'(x_k)}.$$

- By using $\beta = 1$, another scheme is found, whose second step is calculated as follows

$$x_{k+1} = y_k - \frac{6f'(x_k)^2-13f'(x_k)f'(y_k)+3f'(y_k)^2}{2f'(x_k)(3f'(x_k)-5f'(y_k))}\frac{f(x_k)}{f'(x_k)}.$$

- When $\beta = -\frac{3}{2}$, the expression of the second step is

$$x_{k+1} = y_k - \frac{23f'(x_k)^2-24f'(x_k)f'(y_k)+9f'(y_k)^2}{8f'(x_k)^2}\frac{f(x_k)}{f'(x_k)}.$$

- If, for example, $\beta = \frac{1}{2}$, then

$$x_{k+1} = y_k - \frac{(f'(x_k)-3f'(y_k))(5f'(x_k)-f'(y_k))}{8f'(x_k)(f'(x_k)-2f'(y_k))}\frac{f(x_k)}{f'(x_k)}.$$

Another interesting aspect of this class of method is that it can be directly extended to nonlinear systems $F(x) = 0$, as only first derivatives of the nonlinear functions appear in the denominators of the iterative scheme (as it happens with Jarratt's method). Then, the corresponding iterative expression is

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= x^{(k)} - \left(I - \frac{3}{4}\left[I + (\frac{3}{2}+\beta)u^{(k)}\right]^{-1}\left(u^{(k)} + \beta {u^{(k)}}^2\right)\right)[F'(x^{(k)})]^{-1}F(x^{(k)}), \end{aligned} \tag{6}$$

where $u^{(k)} = [F'(x^{(k)})]^{-1}[F'(y^{(k)}) - F'(x^{(k)})]$.

In the following section, some test on different scalar functions will allow us to select the most stable elements of the family and they will be used to estimate the solution of Burgers' partial differential equation in the multidimensional case.

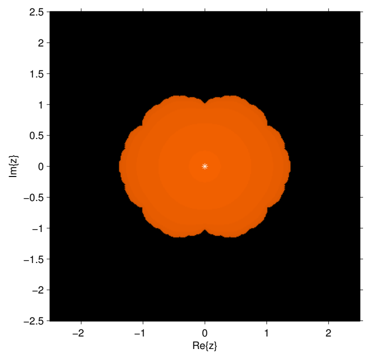
3 Numerical results

In this section, we compare some of the schemes described with the well-known Newton's procedure, that has second-order of convergence. Specifically, we compare it with the elements of the defined class of iterative methods (1) (with equi-spaced values of the parameter β between -2 and 2) with Newton's and Ostrowski's methods.

This comparison will be made, at a first stage, by using dynamical tools: we will use the software described in [3] in order to draw the dynamical planes associated to each one of the members of our proposed class of methods on some specific nonlinear functions. We will try to deduce from the observed behavior of the methods which elements of the family are more stable and reliable.

All these methods will be employed to solve some nonlinear equations:

- $f_1(x) = \arctan(x), \quad \alpha = 0,$
- $f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad \alpha \approx -1.2076478,$
- $f_3(x) = \sin^2 x - x^2 + 1, \quad \alpha \approx \pm 1.4044916,$
- $f_4(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3, \quad \alpha \approx 2.331968,$
- $f_5(x) = (x-1)^3 - 1, \quad \alpha = 2.$

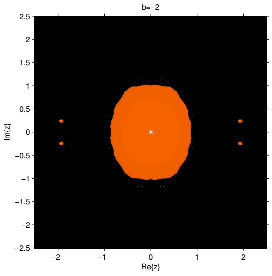


(a) Newton

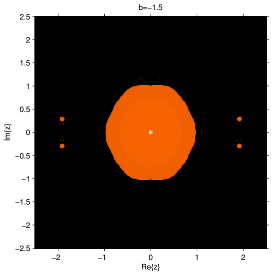
Figure 1: Dynamical plane of Newton's method on $f_1(x) = \arctan(x)$

For the representation of the convergence basins of the procedures, we draw a mesh with four hundred points per axis; each point of the mesh is a different initial estimation which we introduce in each scheme. If the method reaches approximately a solution in less than eighty iterations, this point is drawn in orange (in green, blue,... for other solutions of the nonlinear function). The color will be more intense when the number of iterations is lower. Otherwise, if the method arrives at the maximum of iterations without converging to any solution, the point will be drawn in black.

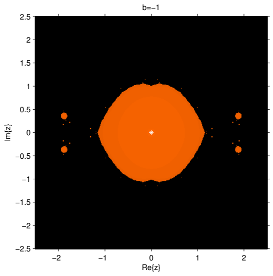
In Figures 1 and 2 the dynamical planes of Newton's and proposed methods on $f_1(x)$ are showed in the region $[-2.5, 2.5] \times [-2.5, 2.5]$. As there exists only one solution, we focus our attention on the wideness of the region of convergence of the different methods. It can be observed that the widest real interval of convergence $[-1.9, 1.9]$ corresponds to $\beta = 1$ (Figure 2e) and $\beta = 2$ (Figure 2f), followed by $\beta = 0$ (Figure 2d), with $[-1.88, 1.88]$ as real region of convergence. These methods highly improve the behavior of Newton's method, not only in the order, but also in the amplitude of the region of starting points.



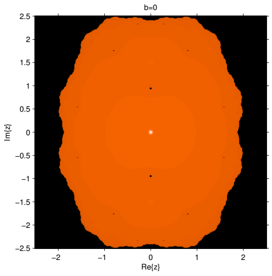
(a) $\beta = -2$



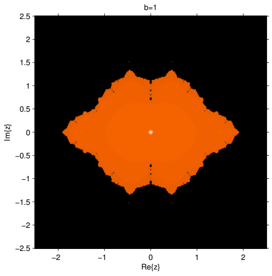
(b) $\beta = -\frac{3}{2}$



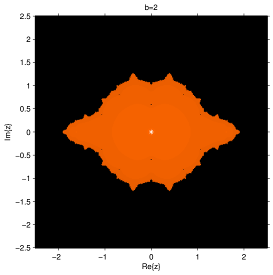
(c) $\beta = -1$



(d) $\beta = 0$

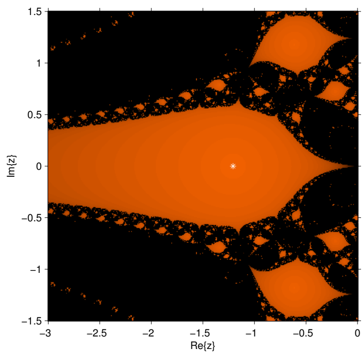


(e) $\beta = 1$



(f) $\beta = 2$

Figure 2: Dynamical planes of the proposed methods on $f_1(x) = \arctan(x)$

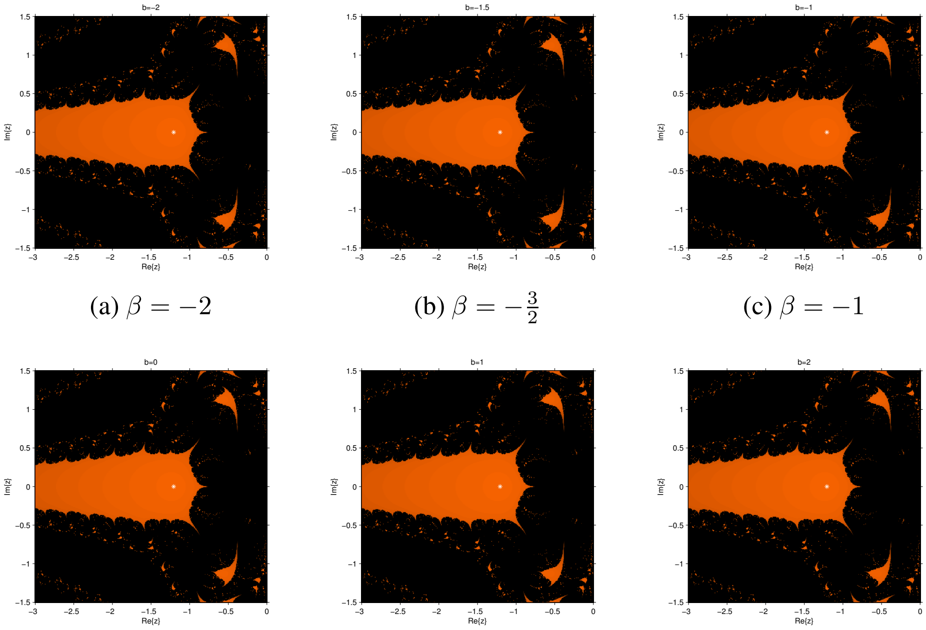


(a) Newton

Figure 3: Dynamical planes of Newton's method on $f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$

In case of function $f_2(x)$, it is observed in Figures 3 and 4 that the best behavior, in terms of stability, corresponds to Newton's method (Figure 3). However, there is not a great difference, in terms of wideness of the basin of convergence to the root, between the proposed elements of the family and classical Newton's method. Indeed, there is a big similitude among the behavior of the elements of the class.

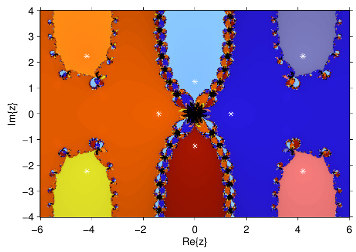
When the real two roots of $f_3(x) = \sin^2 x - x^2 + 1$ are estimated by using Newton's and new methods in $[-2, 2] \times [-2, 2]$, some conclusions can be stated. Firstly, eight different basins of attraction appear (Figures 5 and 6), which is due to the existence of another six complex roots. Taking them into account, the black regions correspond



(a) $\beta = -2$ (b) $\beta = -\frac{3}{2}$ (c) $\beta = -1$

(d) $\beta = 0$ (e) $\beta = 1$ (f) $\beta = 2$

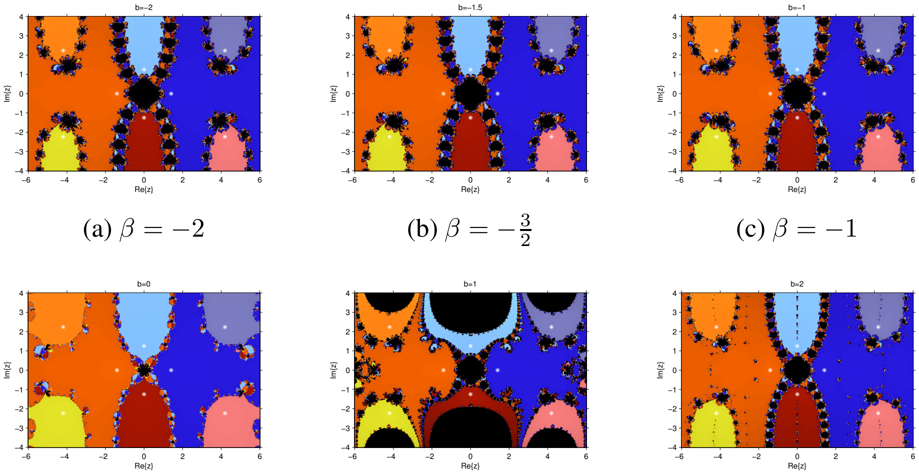
Figure 4: Dynamical planes of the proposed methods on $f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$



(a) Newton

Figure 5: Dynamical planes of Newton's method on $f_3(x) = \sin^2 x - x^2 + 1$

to the pre-images of the infinity, that is, diverging behavior. In terms of stability, the best methods are the member of the proposed family (1) with $\beta = 0$ (see Figure 6d) and Newton's scheme. There are not big differences related to the wideness of the real basins of convergence. In these terms, all the elements are reliable and comparable with Newton's, but with fourth-order convergence.



(a) $\beta = -2$ (b) $\beta = -\frac{3}{2}$ (c) $\beta = -1$

(d) $\beta = 0$ (e) $\beta = 1$ (f) $\beta = 2$

Figure 6: Dynamical planes of the proposed methods on $f_3(x) = \sin^2 x - x^2 + 1$

3.1 Numerical estimation of the solution of Burgers' equation

Now, let us consider the one-dimensional Burgers' partial differential equation (see [1] and [2]),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial^2 x}, \quad (x,t) \in \Omega \tag{7}$$

where $\Omega = (0,1) \times (0,T]$, with initial condition $u(x,0) = f(x)$, $0 < x < 1$ and boundary conditions $u(0,t) = g_1(t)$, $u(1,t) = g_2(t)$, $0 \leq t \leq T$, being Re the Reynolds number and f, g_1 and g_2 are sufficiently smooth given functions.

Burgers' equation appears in turbulence problems, in the theory of shock waves and in continuous stochastic processes. It has applications in gas dynamics, heat conduction, elasticity, etc.

Moreover, Burgers' equation (7) is one of the very few nonlinear partial differential equations which can be solved exactly. The so-called Hopf-Cole transformation [5]

$$u(x,t) = -\left(\frac{2}{Re}\right)\frac{\phi_x(x,t)}{\phi(x,t)},$$

gives us a solution of Burgers' equation (7) if ϕ is a solution of the linear diffusion equation

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial^2 x}.$$

In the following, we will define an implicit finite difference scheme for Burgers' equation, by using $g_1(t) = g_2(t) = 0$ and $f(x) = \frac{2D\beta\pi\sin\pi x}{\alpha+\beta\cos\pi x}$, where $D = \frac{1}{Re} = 0.05$, $\alpha = 5$ and $\beta = 4$. In this difference scheme, a mesh of 51×51 nodes in (x,t) space is considered. With such a discretization, we will obtain a nonlinear system of equations (per instant) $F(u(x,t_j)) = 0$, to be solved by using Newton's scheme and some elements of our proposed family of iterative methods. The approximated value of the solution $u(x,t)$ in each one of these nodes x_i at the instant t_j is located at the (i,j)-entry of the solution matrix U.

Moreover, some numerical computations with proposed functions have been carried out using variable precision arithmetic, with 100 digits, in Matlab 7.11.0. The stopping criterion used at any instant t_j is

$$\|u^{(k+1)}(x,t_j) - u^{(k)}(x,t_j)\| + \|F(u^{(k+1)}(x,t_j))\| < 10^{-30}$$

although both norms will be showed in Table 1 for $t=1$. In any case, for the different methods used, the mean number of iterations $\overline{k}$ (taking into account all the columns of U) also appears. Moreover, some graphics showing the maximum error per instant and the estimated solution will be showed, for some of the iterative methods used. Also the approximate computational order of convergence ρ will appear, according to (see [4])

$$p \approx \rho = \frac{\ln\left(\|u^{(k+1)}(x,t_j) - u^{(k)}(x,t_j)\| / \|u^{(k)}(x,t_j) - u^{(k-1)}(x,t_j)\|\right)}{\ln\left(\|u^{(k)}(x,t_j) - u^{(k-1)}(x,t_j)\| / \|u^{(k-1)}(x,t_j) - u^{(k-2)}(x,t_j)\|\right)}.$$

The value of ρ that is presented in Table 1 is the last coordinate of vector ρ when the variation between its values is small.

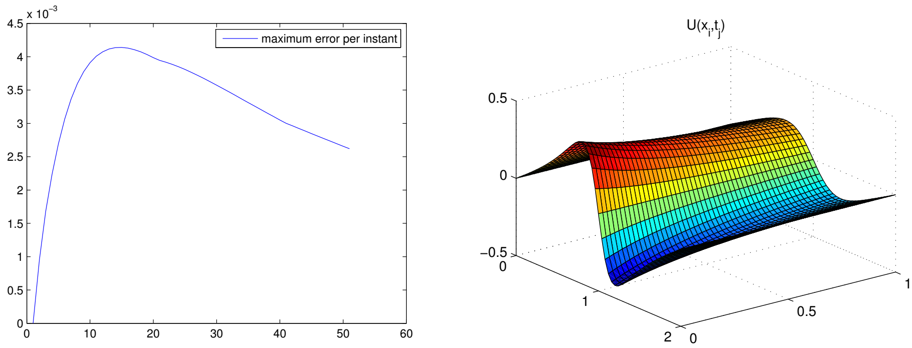
Method	$\|u^{(k+1)}(x,t_j) - u^{(k)}(x,t_j)\|$	$\|F(u^{(k+1)}(x,t_j))\|$	$\overline{k}$	ρ
Newton	1.9004e-59	7.9382e-62	5	1.905904
$\beta=-3/2$	1.0511e-55	5.1320e-58	3	3.934506
$\beta=0$	1.9004e-59	7.9382e-62	3	3.897573
$\beta=0.5$	6.2407e-59	1.7594e-61	3	3.941980
$\beta=1$	3.3192e-57	1.2700e-59	3	3.941884

Table 1: Numerical results for Burgers' equation

Let us remark that the obtained estimations have been compared with the exact (up to 15 digits) solution at the same nodes, showing all the methods the same maximum exact error $MEE = 0.004140236998958$ (see Figure 7), per column of the solution matrix U, what is reasonable being all of them of the same order and taking into account that the error of the discretization process is of second order. From Table 1, we deduce that, in terms of accuracy of the estimated solution, the members of the class of iterative methods whose value of the parameter is close to zero are more precise, what is in concordance with the dynamical results. The approximated computational order of convergence ρ is also around the expected values.

4 Conclusions

A new family of fourth-order of convergence has been presented, fully extensible to nonlinear systems of equations. A dynamical analysis on some interesting functions



(a) Maximum error

(b) $U(x_i, t_j)$, $i, j = 1, 2, \dots, 51$

Figure 7: Maximum error per instant and estimated solution

allows us to select some good members of the family and to apply them on one of the few nonlinear problems that have a known exact solution in applied problems: Burgers' partial differential equation. We have showed how some of the proposed methods behave on this problem and have analyzed the exact error.

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